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On the Simultaneous Distribution of the Fractional Parts of Different Powers of Prime Numbers[1]

Wenguang Zhai

Department of Mathematics, Shandong Normal University, Jinan 250014, China
E-mail: wgzhai@jn-public.sd.cninfo.net

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In this paper we study the distribution modulo 1 of the sequence of vectors

1. INTRODUCTION AND MAIN RESULTS

In 1940 I. M. Vinogradov considered the distribution of the fractional parts of the sequence $f\sqrt{p}$, where p runs over prime numbers (see [15]). This celebrated work motivated the interests of many authors to investigate the distribution of p^θ modulo 1 by various methods (see [1–3, 7, 8, 10–12]).

In [13], Tolev studied the simultaneous distribution of the fractional parts of different powers of prime numbers. Let $k \geqslant 2$ be a fixed integer and $0 < \alpha_k < \cdots < \alpha_1 < 0$ real numbers; $\Gamma \subset R^k$ is defined by

$$\Gamma = \Gamma(\xi_1, \eta_1, ..., \xi_k, \eta_k) = \{(x_1, ..., x_k) \mid \xi_i < x_i < \eta_i, 1 \leqslant i \leqslant k\}$$

where $0 < \xi_i < \eta_i \leqslant 1$, $1 \leqslant i \leqslant k$. Let $\mu(\Gamma) = \prod_{i=1}^k (\eta_i - \xi_i)$, and let $S(x; \Gamma)$ denote the number of primes not greater than x and satisfy the condition

$$(\{p^{\alpha_1}\}, \{p^{\alpha_2}\}, ..., \{p^{\alpha_k}\}) \in \Gamma.$$

Then Tolev proved that

$$S(x; \Gamma) = \pi(x)(\mu(\Gamma) + O(x^{-\delta/3} \log^{k+9} x)) \tag{1.1}$$

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with

$$\delta = \min(1-\alpha_1, \alpha_1-\alpha_2, ..., \alpha_{k-1}-\alpha_k, \alpha_k, 1/4).$$

The aim of this paper is to further improve Tolev's result by different methods. We first prove the following

THEOREM 1. *We have*

$$S(x; \Gamma) = \pi(x)(\mu(\Gamma) + O(x^{-\delta_1} \log^{k+11.5} x)), \tag{1.2}$$

where

$$\delta_1 = \min(1-\alpha_1, \alpha_1-\alpha_2, ..., \alpha_{k-1}-\alpha_k, \alpha_k/3, 20/177).$$

EXAMPLE 1. Take $k=2$. If $80/177 < \alpha_1 < 157/177$, $60/177 < \alpha_2 < \alpha_1 - 20/177$, then

$$S(x; \Gamma) = \pi(x)\, \mu(\Gamma) + O(x^{157/177} \log^{14.5} x). \tag{1.3}$$

Similar to Theorem 1, we can prove the following.

THEOREM 2. *We have*

$$S(x; \Gamma) = \pi(x)(\mu(\Gamma) + O(x^{-\delta_2} \log^{k+11.5} x)), \tag{1.4}$$

where

$$\delta_2 = \min(\alpha_1-\alpha_2, ..., \alpha_{k-1}-\alpha_k, \alpha_k/3, 40/407).$$

EXAMPLE 2. Take $k=2$. If $120/407 < \alpha_2 < \alpha_1 - 40/407$, then

$$S(x; \Gamma) = \pi(x)\, \mu(\Gamma) + O(x^{367/407} \log^{14.5} x). \tag{1.5}$$

Obviously, Theorem 2 is better than Theorem 1 if α_1 is very close to 1.

It is obvious that Theorem 1 and Theorem 2 are very weak if $\delta_0 = \min(\alpha_1-\alpha_2, \alpha_2-\alpha_3, ..., \alpha_{k-1}-\alpha_k)$ is very small. So we shall use a different approach to prove the following Theorem 3, which is better than Theorem 1 and Theorem 2 if δ_0 is small.

THEOREM 3. *We have*

$$S(x; \Gamma) = \pi(x)(\mu(\Gamma) + O(x^{-\delta_3} \log^{k+5.5} x)), \tag{1.6}$$

where

$$\delta_3 = \min(1/(4k+6), \alpha_k/(4k-2)).$$

EXAMPLE 3. Take $k=2$. If $6/14<\alpha_2<\alpha_1<1$, $\alpha_1-\alpha_2<1/14$. Then Theorem 3 yields

$$S(x;\Gamma)=\pi(x)\,\mu(\Gamma)+O(x^{13/14}\log^{8.5}x). \tag{1.7}$$

Notations. $\{l\}$ denotes the fractional part of t. $h=(h_1,...,h_k)$ denotes the k-dimensional vector with integer components.

$$\|h\|=\max_{1\leqslant i\leqslant k}|h_i|; \qquad r(h)=\prod_{i=1}^{k}\max(|h_i|,1).$$

$\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in R^k. $e(x)=e^{2\pi ix}$. $m\sim M$ means $M<m\leqslant 2M$; $m\asymp M$ means $c_1M\leqslant m\leqslant c_2M$ for positive constants c_1 and c_2. $\Lambda(n)$ is the Mangoldt function.

2. SOME PRELIMINARY LEMMAS

We need the following lemmas.

LEMMA 2.1. *Suppose $f(n)$ is a real-valued function on the interval $[N,N_1]$, where $2\leqslant N<N_1\leqslant 2N$. If $0<c_1\lambda_1\leqslant|f'(n)|\leqslant c_2\lambda_1\leqslant 1/2$, then*

$$\sum_{N<n\leqslant N_1}e(f(n))\ll\lambda_1^{-1}.$$

If $|f^{(j)}(n)|\sim\lambda_1N^{-j+1}$ $(j=1,2)$, then

$$\sum_{N<n\leqslant N_1}e(f(n))\ll\lambda_1^{-1}+N^{1/2}\lambda_1^{1/2}.$$

If $|f^{(j)}(n)|\sim\lambda_1N^{-j+1}$ $(j=1,2,3,4,5,6)$, then

$$\sum_{N<n\leqslant N_1}e(f(n))\ll\lambda_1^{-1}+N^{\lambda}\lambda_1^{\kappa}.$$

Where (κ,λ) is any exponent pair.

Proof. See Graham and Kolesnik [6].

LEMMA 2.2. *Suppose that $0<a<b\leqslant 2a$ and R is an open convex set in C containing the real segment $[a,b]$. Suppose further that $f(z)$ is analytic on R, $f(x)$ is real for real x in R, $|f''(z)|\leqslant M$ for $z\in R$, and there is a constant*

$k>0$ such that $f''(x)\leqslant -kM$ for all real x in R. Let $\alpha=f'(b)$ and $\beta=f'(a)$, and define x_v for each integer v in the range $\alpha<v<\beta$ by $f'(x_v)=v$. Then we have

$$\sum_{a<n\leqslant b} e(f(n))=e(-\tfrac{1}{8})\sum_{\alpha<v\leqslant\beta}|f''(x_v)|^{-1/2}e(f(x_v)-vx_v)$$
$$+O(M^{-1/2}+\log(2+M(b-a))).$$

Proof. This is Lemma 6 of Heath-Brown [9].

LEMMA 2.3. *Suppose $z(n)$ is any complex number, $1\leqslant Q\leqslant N$, then*

$$\Big|\sum_{N<n\leqslant CN} z(n)\Big|^2 \ll \frac{N}{Q}\sum_{0\leqslant q\leqslant Q}\left(1-\frac{q}{Q}\right) Re \sum_{N<n\leqslant CN-q} z(n)\,\overline{z(n+q)}.$$

Proof. This is Lemma 2.5 of Graham and Kolesnik [6].

LEMMA 2.4. *Let $0<L\leqslant N<vN\leqslant\lambda L$ and a_n be complex numbers with $|a_n|\leqslant 1$. Then we have*

$$\sum_{N<n\leqslant vN} a_n=\frac{1}{2\pi}\int_{-L}^{L}\left(\sum_{L<l\leqslant L} a_l l^{-it}\right)\frac{(vN)^{it}-N^{it}}{t}\,dt+O(\log(2+L)).$$

Proof. This is Lemma 6 of Fouvry and Iwaniec [5].

LEMMA 2.5. *If $Z_n=(Z_{1,n}, ..., Z_{k,n})$ $(n=1, 2, 3, ...)$ is a sequence of k-dimensional vectors and*

$$D_N=\sup_{\Gamma}\left|\frac{1}{N}\sum_{\substack{n\leqslant N\\(Z_{1,n}, ..., Z_{k,n})}} 1-\mu(\Gamma)\right|$$

is its discrepency (the supremum is taken over the set of all Γ of type (1.1)), then the inequality

$$D_N\ll\frac{1}{M}+\sum_{0<\|m\|\leqslant M}\frac{1}{r(m)}\left|\frac{1}{N}\sum_{n\leqslant N} e(\langle m, Z_n\rangle)\right|$$

holds. (M is an arbitrary positive number; the constant inplied in the $\ll$ symbol depends only on k.)

Proof. This is Lemma 3 of Tolev [13].

LEMMA 2.6. *Let $\mathscr{X}$ and $\mathscr{Y}$ be two finite set of real numbers, $\mathscr{X} \subset [-X, X]$, and $\mathscr{Y} \subset [-Y, Y]$. Then for any complex function $u(x)$ and $v(y)$ we have*

$$\left| \sum_{x \in \mathscr{X}} \sum_{y \in \mathscr{Y}} u(x)\, v(y)\, e(xy) \right|^2$$

$$\leqslant 20(1+XY) \sum_{\substack{x,\, x' \in \mathscr{X} \\ |x-x'| \leqslant Y^{-1}}} |u(x)\, u(x')| \sum_{\substack{y,\, y' \in \mathscr{Y} \\ |y-y'| \leqslant X^{-1}}} |v(y)\, v(y')|.$$

Proof. This is the one-dimensional case of Lemma 2.5 of [4]; see also Proposition 1 of Fouvry and Iwaniec [5].

LEMMA 2.7. *Let u and v be positive real numbers. If $n > v$ then*

$$\Lambda(n) = -\sum_{\substack{n=kl \\ k>v,\, l>u}} \Lambda(k) \sum_{d \mid l,\, d \leqslant u} \mu(d) + \sum_{n=kl,\, l \leqslant u} \mu(k) \log l$$

$$- \sum_{\substack{n=klm \\ l \leqslant u,\, m \leqslant v}} \Lambda(k)\, \mu(m).$$

Proof. See Vaughan [14].

Suppose $d \geqslant 2$ is a fixed integer; $a_1, a_2, ..., a_d$ are any real numbers with $a_1 a_2 \cdots a_d \neq 0$; $\gamma_1, ..., \gamma_d$ are real noninteger constants; and M and M_1 are real numbers such that $5 < M < M_1 \leqslant 2M$. Let

$$f_d(m) = a_1 m^{\gamma_1} + \cdots + a_d m^{\gamma_d}.$$

Set $R = |a_1|\, M^{\gamma_1} + \cdots + |a_d|\, M^{\gamma_d}$. For the exponential sum

$$S_d(m) = \sum_{M < m \leqslant M_1} e(f_d(m)),$$

we have the following two lemmas, which are Propositions 2 and 3 of Zhai [16] respectively.

LEMMA 2.8. *If $R \leqslant \eta M$, where η is a fixed positive constant small enough, then*

$$S_d(M) \ll MR^{-1/d}.$$

LEMMA 2.9. *We have*

$$S_d(M) \ll R^{1/2} + MR^{-1/(d+1)}.$$

3. ESTIMATION OF AN EXPONENTIAL SUM OVER PRIMES

Suppose Y is a large positive real number; $0 < \alpha < 1$; $0 < \delta(\alpha) < 1/3$ is a function of α; h is an integer such that $1 \leqslant h \ll Y^\delta$; and $g(m)$ is a real function on $[Y, 2Y]$ of the form

$$g(m) = u_1 m^{\gamma_1} + \cdots + u_l m^{\gamma_l}$$

such that $|g^{(j)}(m)| \leqslant \varepsilon h Y^{\alpha - j}$ ($j = 0, 1, 2, ..., 6$), where $l \geqslant 1$, $\gamma_1, ..., \gamma_l$ are real constants; $u_1, ..., u_l$ are any real numbers; and ε is a sufficiently small positive constant. The aim of this section is to estimate the exponential sum over primes

$$S(Y; h, \alpha) = \sum_{Y < m \leqslant 2Y} \Lambda(m)\, e(hm^\alpha + g(m)).$$

For convenience, in this section we always set $F = hY^\alpha$. The constants implied by $\ll$ in this section depend only on α, $\gamma_1, ..., \gamma_l$ and ε.

LEMMA 3.1. *Suppose* $340/531 < \alpha < 1$, $\delta = \delta(\alpha) = \min(1 - \alpha, 20/177)$, *and* $0 < \Delta \leqslant \delta$. *Let* a_m *be a sequence of complex numbers such that*

$$\sum_{m \sim M} |a_m|^2 \ll M \log^{2A} M, \qquad A > 0.$$

Then for $h \ll Y^\delta$, $M \ll Y^{1-2\Delta}$, *we have*

$$S_l = \sum_{m \sim M} a(m) \sum_{Y < mn \leqslant 2Y} e(h(mn)^\alpha + g(mn)) \ll Y^{1-\Delta} \log^{A+1} Y. \quad (3.1)$$

Proof. Write $f(m) = hm^\alpha + g(m)$ and let $N = Y/M$.
By Lemma 2.1 with the exponent pair $(2/7, 4/7)$ we get

$$\begin{aligned} S_I &\ll \sum_{m \sim M} |a_m|\, (NF^1 + F^{2/7} N^{2/7}) \\ &\ll MNF^{-1} \log^A Y + MF^{2/7} N^{2/7} \log^A Y \\ &\ll Y^{1-\alpha} \log^A Y + Y^{1-\Delta} \log^A Y \ll Y^{1-\Delta} \log^A Y, \end{aligned} \quad (3.2)$$

if $M \ll Y^{(5-7\Delta)/5} F^{-2/5}$.

Now suppose $M \gg Y^{(5-7\Delta)/5} F^{-2/5}$. Take $Q = [Y^{2\Delta} \log^{-1} Y]$, then $Q = o(N)$. By Cauchy's inequality and Lemma 2.3 we get

$$|S_I|^2 \ll \sum_{m \sim M} |a_m|^2 \sum_{m \sim M} \left| \sum_{n \sim Y/m} e(f(mn)) \right|^2$$
$$\ll \frac{M^2 N^2 \log^{2A} M}{Q} + \frac{MN \log^{2A} M}{Q} \sum_{q=1}^{Q} |E_q|, \tag{3.3}$$

where

$$E_q = \sum_{M < m \leqslant 2M} \sum_{Y/m < n \leqslant 2Y/m - q} e\left(hm^{\alpha} \Delta(n, q; \alpha) + \sum_{j=1}^{l} u_j m^{\gamma_j} \Delta(n, q; \gamma_j) \right),$$
$$\Delta(n, q; t) = (n+q)^t - n^t.$$

So the problem is reduced to show that

$$\sum_{q=1}^{Q} |E_q| \ll Y. \tag{3.4}$$

For fixed q, write

$$G(m, n) = G(m, n; q) = hm^{\alpha} \Delta(n, q; \alpha) + \sum_{j=1}^{l} u_j m^{\gamma_j} \Delta(n, q; \gamma_j).$$

If $\max |\partial G/\partial m| \leqslant 1/2$, then by Lemma 2.1 we get

$$E_q \ll \frac{MN^2}{Fq}. \tag{3.5}$$

If $|\partial G/\partial n| \leqslant 1/2$, we can get the same estimate.

Now suppose $|\partial G/\partial m| \gg 1$, $|\partial G/\partial n| \gg 1$. We have

$$E_q = \sum_{Y/2M < n \leqslant 2Y/M - q} \sum_{M_1 < m \leqslant M_2} e(G(m, n)),$$

where $M_1 = \max(M, Y/n)$, $M_2 = \min(2M, 2Y/(n+q))$. Using Lemma 2.2 we have

$$\sum_{M_1 < m \leqslant M_2} e(G(m, n))$$
$$= c \sum_{r_1(n) < r \leqslant r_2(n)} \left| \frac{\partial^2 G}{\partial m^2}(m_r, n) \right|^{-1/2} e(G(m_r, n) - rm_r)$$
$$+ O(\log Y + MN^{1/2} (Fq)^{-1/2}),$$

where $m_r = m(r, n)$ is the solution of the equation $(\partial G/\partial m)(m, n) = r$ and

$$r_1(n) = \frac{\partial G}{\partial m}(M_1, n), \qquad r_2(n) = \frac{\partial G}{\partial m}(M_2, n).$$

Let $r_1 = \min_n r_1(n)$, $r_2 = \min_n r_2(n)$, and

$$\varphi(n, r) = \frac{(Fq)^{1/2}}{MN^{1/2}} \left| \frac{\partial^2 G}{\partial m^2}(m_r, n) \right|^{-1/2}, \qquad s(r, n) = G(m_r, n) - rm_r.$$

For each $r_1 \leqslant r \leqslant r_2$, the equations

$$Y/2M < n \leqslant 2Y/M - q, \qquad r_1(n) < r \leqslant r_2(n)$$

define an interval $I(r) = (n_1(r), n_2(r)]$.

So we have

$$E_q \ll \frac{MN^{1/2}}{(Fq)^{1/2}} \sum_{r_1 \leqslant r \leqslant r_2} \left| \sum_{n \in I(r)} \varphi(n, r)\, e(s(r, n)) \right| + N \log Y + MN^{3/2}(Fq)^{-1/2},$$

where $I(r) = (n_1(r), n_2(r)]$.

We can verify that

$$\varphi(n, r) \ll 1, \qquad \varphi'(n, r) \ll N^{-1},$$

$$\frac{\partial^j s}{\partial n^j}(r, n) \sim FqN^{-j-1}, \qquad j = 0, 1, 2, 3, 4, 5, 6.$$

So by Lemma 2.1 with the exponent (2/18, 13/18) we get

$$\begin{aligned} E_q &\ll \frac{MN^{1/2}}{(Fq)^{1/2}} \frac{Fq}{MN} \left(\frac{Fq}{N^2}\right)^{2/18} N^{13/18} \\ &\quad + N \log Y + MN^{3/2}(Fq)^{-1/2} \\ &\ll (Fq)^{11/18} + N \log Y + MN^{3/2}(Fq)^{-1/2}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) we have

$$\sum_{q=1}^{Q} |E_q| \ll \frac{MN^2 \log Q}{F} + F^{11/18} Q^{29/18} + NQ \log Y + MN^{3/2} Q^{1/2} (F)^{-1/2} = \sum_{j=1}^{4} I_j.$$

Since $\alpha \geqslant 340/531$, we have $\Delta \leqslant \delta \leqslant 3\alpha/17$, which gives $M \gg Y^{(5-7\Delta)/5} F^{-2/5} \gg YQ/F$. So we have $I_1 \ll Y$, $I_4 \ll Y$. Since $11\alpha + 69\delta \leqslant 18$, we have $I_2 \ll Y$. Since $2\alpha + 19\delta < 5$, we have $M \gg Y^{(5-7\Delta)/5} F^{-2/5} \gg Q \log Y$, so $I_3 \ll Y$. Combining the above, we get (3.4). This completes the proof of Lemma 3.1.

Lemma 3.2. *Suppose* $340/531 < \alpha < 1$, $\delta = \delta(\alpha) = \min(1-\alpha, 20/177)$, $0 < \Delta \leqslant \delta$. *Let* $a(m)$ *and* $b(n)$ *be complex numbers such that*

$$\sum_{M < m \leqslant 2M} |a(m)|^2 \ll M \log^{2A} M,$$

$$\sum_{N < n \leqslant 2N} |b(n)|^4 \ll N \log^{4B} N, \qquad A > 0, \quad B > 0.$$

Then for $Y^{2\Delta} \ll N \ll \min(FY^{-2\Delta}, Y^{1-4\Delta}, Y^{(93-318\Delta)/9} F^{-53/9})$, $MN \sim Y$, *we have*

$$\begin{aligned} S_{\mathrm{II}} &= \sum_{m \sim M} a(m) \sum_{n \sim N} b(n)\, e(h(mn)^\alpha + g(mn)) \\ &\ll Y^{1-\Delta} \log^{A+B+1} Y. \end{aligned} \tag{3.7}$$

Proof. Take $Q = [Y^{2\Delta} \log^{-1} Y]$, then $Q = o(N)$. By Cauchy's inequality and Lemma 2.3 we get

$$\begin{aligned} |S_{\mathrm{II}}|^2 &\ll \sum_{m \sim M} |a(m)|^2 \sum_{m \sim M} \left| \sum_{n \sim N} b(n)\, e(f(mn)) \right|^2 \\ &\ll \frac{M^2 N^2 \log^{2A+2B} Y}{Q} + \frac{MN \log^{2A} Y}{Q} \sum_{q=1}^{Q} E_q, \end{aligned} \tag{3.8}$$

where

$$E_q = \sum_{n \sim N} |b(n+q)\, b(n)| \left| \sum_{m \sim M} e(G(m,n)) \right|$$

and $G(m, n)$ is defined in the same way as in the proof of Lemma 3.1.

If $|\partial G/\partial m| \leqslant 10^3 M q^{-2}$, by Lemma 2.1 we get

$$\begin{aligned} E_q &\ll \sum_{n \sim N} |b(n+q)\, b(n)| \left(\frac{MN}{Fq} + \left(\frac{Fq}{MN} \right)^{1/2} M^{1/2} \right) \\ &\ll \sum_{n \sim N} (|b(n+q)|^2 + |b(n)|^2) \left(\frac{MN}{Fq} + M/q \right) \\ &\ll MN/q \log^{2B} Y, \end{aligned} \tag{3.9}$$

if we note that $N \ll F$.

Now suppose $|\partial G/\partial m| > 10^3 M q^{-2}$. By Lemma 2.2 we get

$$\sum_{m\sim M} e(G(m,n)) \ll \frac{MN^{1/2}}{(Fq)^{1/2}} \left| \sum_{r_1(n)\leqslant r\leqslant r_2(n)} \varphi(n,r)\, e(s(r,n)) \right| + \log Y + MN^{1/2}(Fq)^{-1/2},$$

where $s(r,n)$, $\varphi(n,r)$ are defined in the same way as in the proof of Lemma 3.1 and

$$r_1(n) = \frac{\partial G}{\partial m}(M,n), \qquad r_2(n) = \frac{\partial G}{\partial m}(2M,n).$$

Thus we have

$$E_q \ll \frac{MN^{1/2}}{(Fq)^{1/2}} \sum_{n\sim N} |b(n+q)\, b(n)| \left| \sum_{r_1(n)<r\leqslant r_2(n)} \varphi(n,r)\, e(s(r,n)) \right| + N\log^{2B+1} Y + MN^{3/2}(Fq)^{-1/2}\log^{2B} Y. \tag{3.10}$$

So it suffices to bound the sum

$$\Sigma_1 = \sum_{n\sim N} |b(n+q)\, b(n)| \left| \sum_{r_1(n)<r\leqslant r_2(n)} \varphi(n,r)\, e(s(r,n)) \right|.$$

By Cauchy's inequality and Lemma 2.3 again we get

$$\begin{aligned}\Sigma_1^2 &\ll \sum_{n\sim N} |b(n+q)\, b(n)|^2 \sum_{n\sim N} \left| \sum_{r_1(n)<r\leqslant r_2(n)} \varphi(n,r)\, e(s(r,n)) \right|^2 \\ &\ll \frac{N^2R^2\log^{4B} Y}{T} + \frac{NR\log^{4B} Y}{T}\Sigma_2\end{aligned} \tag{3.11}$$

with $T = [Fq^3/M^2N]$ and $R = Fq/MN$, where

$$\Sigma_2 = \sum_{t=1}^{T} \left| \sum_{n\sim N} \sum_{r_1(n)<r\leqslant r_2(n)-t} \varphi(n,r+t)\, \varphi(n,r)\, e(s(r+t,n)-s(r,n)) \right|$$

and where we used the estimate

$$\sum_{n\sim N} |b(n+q)\, b(n)|^2 \ll \sum_{n\sim N} (|b(n+q)|^4 + |b(n)|^4) \ll N\log^{4B} Y.$$

It is easy to check that $10 < T = o(R)$.

We recall that $s(r, n) = G(m(r, n), n) - rm(r, n)$, where $m(r, n)$ denotes the solution of

$$\frac{\partial G}{\partial m}(m, n) = r.$$

It can be easily seen that

$$\frac{\partial s}{\partial r}(r, n) = \frac{\partial G}{\partial m}\frac{\partial m}{\partial r} - m(r, n) - r\frac{\partial m}{\partial r} = -m(r, n).$$

So we have

$$\begin{aligned} H(n) := H_{r,t,q}(n) &= s(r+t, n) - s(r, n) \\ &= \int_r^{r+t} \frac{\partial s}{\partial u}(u, n)\, du = -\int_r^{r+t} m(u, n)\, du, \end{aligned}$$

which implies that

$$|H^{(j)}(n)| \sim tMN^{-j} \qquad (j = 0, 1, 2, 3, 4, 5, 6.)$$

Let $I(r, t)$ denote the interval

$$N < n \leqslant 2N, \qquad r_1(n) < r \leqslant r_2(n) - t.$$

Then

$$\Sigma_2 \ll \sum_{t=1}^{T} \sum_{r \sim R} \left| \sum_{n \in I(r,t)} \varphi(n, r+t)\, \varphi(n, r)\, e(s(r+t, n) - s(r, n)) \right|.$$

Thus using partial summation and then using the exponent pair $(13/40, 22/40) = (BA^2)^2\,(1/2, 1/2)$, we get

$$\begin{aligned} \Sigma_2 &\ll \sum_{t=1}^{T} \sum_{r \sim R} (tM/N)^{13/40}\, N^{22/40} \\ &\ll RM^{13/40}N^{9/40}T^{53/40} \ll NR \end{aligned} \tag{3.12}$$

if we note that $N \ll Y^{(93-318\Delta)/9}F^{-53/9}$.

Combining (3.9)–(3.12) we get that for any $1 \leqslant q \leqslant Q$,

$$\begin{aligned} E_q \ll{}& \frac{MN\log^{2B+1} Y}{q} + N\log^{2B+1} Y \\ &+ MN^{3/2}(Fq)^{-1/2}\log^{2B} Y. \end{aligned} \tag{3.13}$$

Now Lemma 3.2 follows from inserting (3.13) into (3.8).

From Lemma 3.1 and Lemma 3.2 we can prove the following Proposition 3.1.

PROPOSITION 3.1. *Suppose* $340/531 < \alpha < 1$, $\delta = \delta(\alpha) = \min(1-\alpha, 20/177)$, $0 < \varDelta \leqslant \delta$. *Then, for* $h \ll Y^{\delta}$, *we have*

$$S(Y; h, \alpha) \ll Y^{1-\varDelta} \log^{11.5} Y. \tag{3.14}$$

Proof. Applying Lemma 2.7 to $S(Y, h, \alpha)$ we get

$$\begin{aligned} S(Y, h, \alpha) = & \sum_{m \leqslant u} \sum_{n \sim Y/m} \mu(m) \log n e(f(mn)) \\ & - \sum_{m \leqslant uv} \sum_{n \sim Y/m} b(m)\, e(f(mn)) \\ & - \sum_{u < n \leqslant 2Y/v} \sum_{m \sim Y/n,\, m > v} a(n)\, \varLambda(m)\, e(f(mn)) \\ = & S_1 - S_2 - S_3, \end{aligned} \tag{3.15}$$

where

$$a(m) = \sum_{d \mid m,\, d \leqslant u} \mu(d), \qquad b(n) = \sum_{n = de,\, d \leqslant u,\, e \leqslant v} \mu(d)\, \varLambda(e).$$

Note that $|a(m)| \leqslant d(m)$ and $|b(m)| \leqslant \sum_{d \mid m} \varLambda(d) = \log m$. So, for any $D \geqslant 2$,

$$\sum_{m \sim D} |a(m)|^2 \ll D \log^3 D, \qquad \sum_{n \sim D} |b(n)|^2 \ll D \log^2 D.$$

It suffices to bound S_i $(i = 1, 2, 3)$. We take $u = Y^{2\varDelta}$, $v = Y/E$, where

$$E = \min(FY^{-2\varDelta}, Y^{1-4\varDelta}, Y^{(93-318\varDelta)/9} F^{-53/9}).$$

Using Lemma 1 with the exponent pair (1/2, 1/2) we get

$$\begin{aligned} S_1 & \ll \sum (Y/mF + F^{1/2}) \log Y \\ & \ll (Y/F \log^2 Y + uF^{1/2} \log Y) \ll Y^{1-\varDelta} \log^2 Y. \end{aligned}$$

S_2 can be divided into $O(\log Y)$ sums of the form

$$S_2(M) = \sum_{m \sim M} b(m) \sum_{n \sim Y/m} e(f(mn)) \qquad (M \ll uv). \tag{3.16}$$

It is easy to show that $uv \ll Y^{1-2\Delta}$. So by Lemma 3.1 each subsum is $\ll Y^{1-\Delta} \log^2 Y$. After adding these subsums, we find that

$$S_2 \ll Y^{1-\Delta} \log^3 Y.$$

S_3 can be divided into $O(\log Y)$ sums of the form

$$S_3(N) = \sum_{n \sim N} \sum_{m \sim Y/n,\, m > v} a(n)\, \Lambda(m)\, e(f(mn)) \qquad (u \ll N \ll Y/v). \tag{3.17}$$

By Lemma 2.4

$$\begin{aligned}\sum_{m \sim Y/n,\, m > v} \Lambda(m)\, e(f(mn)) &= \frac{1}{2\pi} \int_{-M}^{M} \left(\sum_{M < m < 4M} \Lambda(m)\, m^{it} e(f(mn)) \right) \\ &\quad \times \frac{m_2^{it}(n) - m_1^{it}(n)}{t}\, dt + O(\log^2 Y)\end{aligned}$$

with $M = Y/2N$.

So we have

$$\begin{aligned} S_3(N) &= \frac{1}{2\pi} \int_{M}^{M} \sum_{n \sim N} a(n) \sum_{M < m < 4M} \Lambda(m)\, m^{it} e(f(mn)) \\ &\quad \times \frac{m_2^{it}(n) - m_1^{it}(n)}{t}\, dt + O(\log^2 Y) \\ &\ll |S_3^*(N)| \log Y + HN \log^4 Y, \end{aligned} \tag{3.18}$$

where

$$S_3^*(N) = \sum_{n \sim N} \sum_{m \sim M} a^*(n)\, b^*(m)\, e(f(mn)), \tag{3.19}$$

with $a^*(n) \ll d(n)$, $b^*(m) \ll \log m$. So we have

$$\sum_{n \sim N} |a^*(n)|^4 \ll N \log^{15} N, \qquad \sum_{m \sim m} |b^*(m)|^2 \ll M \log^2 M.$$

By Lemma 3.2 we get $S_3^*(N) \ll Y^{1-\Delta} \log^{9.5} Y$. Combining the above we get

$$S_3 \ll Y^{1-\Delta} \log^{11.5} Y.$$

This completes the proof of Proposition 3.1.

By the same proof we can prove the following.

PROPOSITION 3.1*. *Suppose* $340/531 < \alpha < 1$, $\delta = 40/407$, $0 < \Delta \leqslant \delta$. *Then for* $h \ll Y^{\delta}$, *we have*

$$S(Y; h, \alpha) \ll Y^{1-\Delta} \log^{11.5} Y.$$

LEMMA 3.3. *Suppose* $0 < \alpha < 4/5$, $\delta = \delta(\alpha) = \min((1-\alpha)/3, \alpha/4)$, $0 < \Delta \leqslant \delta$. *Let* a_m *be a sequence of complex numbers such that*

$$\sum_{M < m \leqslant 2M} |a_m|^2 \ll M \log^{2A} X, \qquad A > 0.$$

Then for $h \ll Y^{\delta}$, $M \ll YF^{-1}$, *we have*

$$S_I = \sum_{m \sim M} a(m) \sum_{n \sim Y/m} e(h(mn)^{\alpha} + g(mn)) \ll Y^{1-\Delta} \log^4 Y. \tag{3.20}$$

Proof. It immediately follows from Lemma 2.1.

LEMMA 3.4. *Suppose* $0 < \alpha < 4/5$, $\delta = \delta(\alpha) = \min((1-\alpha)/3, \alpha/4)$, $0 < \Delta \leqslant \delta$. *Let* $a(m)$ *and* $b(n)$ *be complex numbers such that*

$$\sum_{M < m \leqslant 2M} |a(m)|^2 \ll M \log^{2A} M,$$

$$\sum_{N < m \leqslant 2N} |b(n)|^2 \ll N \log^{2B} N, \qquad A > 0, \quad B > 0.$$

Then for $h \ll Y^{\delta}$, $Y^{2\Delta} \ll N \ll F$, *we have*

$$\begin{aligned} S_{\mathrm{II}} &= \sum_{m \sim M} a(m) \sum_{n \sim N} b(n)\, e(h(mn)^{\alpha} + g(mn)) \\ &\ll Y^{1-\Delta} \log^{A+B+1} Y. \end{aligned} \tag{3.21}$$

Proof. We begin with (3.8). Since $\Delta \leqslant (1-\alpha)/3$, we have $|\partial G/\partial m| \leqslant 1/2$. By Lemma 2.1 we have

$$\begin{aligned} \sum_{q=1}^{Q} E_q &\ll \sum_{q=1}^{Q} \sum_{n \sim N} |b(n+q)\, b(n)| \frac{MN}{Fq} \\ &\ll MN^2F^{-1} \log^{2B+1} Y \ll Y \log^{2B+1} Y. \end{aligned} \tag{3.22}$$

Whence Lemma 3.4 follows.

PROPOSITION 3.2. *Suppose* $0<\alpha<4/5$, $\delta=\min((1-\alpha)/3, \alpha/4)$, $0<\Delta\leqslant\delta$. *Then for* $h\ll Y^{\delta}$ *we have*

$$S(Y;h,\alpha)\ll Y^{1-\Delta}\log^{5.5} Y. \tag{3.23}$$

Proof. We use notations in the proof of Proposition 3.1. Take $u=Y^{2\Delta}$, $v=Y/F$.

By Lemma 2.1 we have

$$S_1\ll Y^{1-\Delta}\log Y.$$

To estimate S_2, we only need to estimate $S_2(M)$. If $M\ll Y/F=v$, by Lemma 3.3 we get $S_2(M)\ll Y^{1-\Delta}\log^2 Y$. If $v\ll M\ll uv$, then $Y^{2\Delta}\ll N=Y/M\ll F$. So by Lemma 3.4 we can get the same estimate. Thus we have

$$S_2\ll Y^{1-\Delta}\log^3 Y.$$

By Lemma 3.4 we have

$$S_3^*(N)\ll Y^{1-\Delta}\log^{3.5} Y.$$

Thus

$$S_3\ll Y^{1-\Delta}\log^{5.5} Y.$$

This completes the proof of Proposition 3.2.

PROPOSITION 3.3. *Suppose* $0<\alpha<2/3$, $\delta=\min((1-\alpha)/3, \alpha/2, 1/6)$. *Then for* $h\ll Y^{\delta}$, *we have*

$$\sum_{m\sim M}\Lambda(m)\,e(hm^{\alpha})\ll Y^{1-\delta}\log^{4/5} Y. \tag{3.24}$$

Proof. We begin with (3.15) by choosing $u=v=Y^{1/3}$.

By Lemma 2.1 with the exponent pair $(1/2, 1/2)$ we have

$$S_1\ll\sum_{m\leqslant Y^{1/3}}(Y/mF+F^{1/2})\ll YF\log F+Y^{1/3}F^{1/2}$$

$$\ll Y^{1-\alpha}\log Y+Y^{7/9}\ll Y^{1-\delta}\log Y.$$

Now we estimate $S_3^*(N)$ for $Y^{1/3} \ll N \ll Y^{2/3}$. By Lemma 2.6 we have

$$|S_3^*(N)|^2 \ll F\mathscr{A}\mathscr{B},$$

where

$$\begin{aligned}\mathscr{A} &= \sum_{|n_1^\alpha - n_1^\alpha| \leqslant 1/hM^\alpha} |a^*(n_1)\, a^*(n_2)| \ll \sum_{|n_1 - n_2| \ll N/F} d^2(n_1)\\ &\ll \sum_{n_1 \sim N} d^2(n_1) \sum_{|n_2 - n_1| \ll N/F} 1 \ll N(1+N/F)\log^3 N,\\ \mathscr{B} &\ll \sum_{|m_1^\alpha - m_1^\alpha| \leqslant 1/hN^\alpha} |b(m_1)\, b(m_2)| \ll M(1+M/F)\log^2 Y.\end{aligned}$$

So we get

$$\begin{aligned}S^*(N)\log^{-5/2} Y &\ll (FMN)^{1/2} + MN^{1/2} + M^{1/2}N + MN/F^{1/2}\\ &\ll Y^{1-\delta},\end{aligned}$$

which implies

$$S_3 \ll Y^{1-\delta}\log^{4.5} Y.$$

Finally, we estimate S_2. If $M \ll Y^{1/3}$, then similar to S_1 we have $S_2(M) \ll Y^{1-\alpha}\log Y + Y^{7/9}\log Y$. If $Y^{1/3} \ll M$, then similar to $S_3^*(N)$ we have $S_2(M) \ll Y^{1-\delta}\log^2 Y$. Thus we get

$$S_2 \ll Y^{1-\delta}\log^3 Y.$$

This completes the proof of Proposition 3.3.

4. PROOFS OF THEOREM 1 AND THEOREM 2

We only prove Theorem 1. It suffices to establish the inequality

$$R(Y) \ll Y^{-\delta_1}\log^{k+11.5} Y. \tag{4.1}$$

for all $Y \in [x^{1-\delta_1}, x]$, where

$$R(Y) = \sup_{\Gamma}\left|\frac{1}{\pi(2Y)-\pi(Y)}\,(S(2Y;\Gamma) - S(Y;\Gamma)) - \mu(\Gamma)\right|.$$

According to Lemma 2.5 we have

$$\begin{aligned} R(Y) &\ll H^{-1} + \sum_{0<\|h\|\leqslant H} \frac{1}{r(h)} \\ &\quad\times \left| \frac{1}{\pi(2Y)-\pi(Y)} \sum_{Y<p\leqslant 2Y} e(h_1 p^{\alpha_1} + \cdots + h_k p^{\alpha_k}) \right| \\ &\ll H^{-1} + Y^{-1/2} \log^{k+2} Y + Y^{-1} \log Y \sum_{0<\|h\|\leqslant H} \frac{1}{r(h)} |U(h)| \qquad (4.2) \end{aligned}$$

for every $H>2$, where

$$U(h) = \sum_{Y<n\leqslant 2Y} \Lambda(n)\, e(H(n)),$$

$$H(t) = h_1 t^{\alpha_1} + \cdots + h_k t^{\alpha_k}.$$

Propositions from the past section can be applied to estimate $U(h)$. We take $H = CY^{\delta_1}$, where C is a sufficiently large positive constant.

Let $h=(h_1, ..., h_k)$ satisfy $0<\|h\|\leqslant H$ and d be the first integer j with $h_j \neq 0$, then $H(t) = h_d t^{\alpha_d} + g(t)$. Since $\delta_1 \leqslant \alpha_d - \alpha_{d+1}$, we have $g(t) = O(C^{-1} |h_d|\, Y^{\alpha_d})$.

If $\alpha_d > 340/531$, we use Proposition 3.1 to estimate $U(h)$. We take $\Delta = \alpha_d - \alpha_{d+1}$ if $\alpha_d - \alpha_{d+1} \leqslant \min(1-\alpha_d, 20/177)$, and $\Delta = \min(1-\alpha_d, 20/177)$ otherwise. We get

$$\begin{aligned} U(h) &\ll Y^{1-\min(1-\alpha_d,\, \alpha_d-\alpha_{d+1},\, 20/177)} \log^{11.5} Y \\ &\ll Y^{1-\min(1-\alpha_1,\, \alpha_d-\alpha_{d+1},\, 20/177)} \log^{11.5} Y \\ &\ll Y^{1-\delta_1} \log^{11.5} Y. \end{aligned}$$

Now suppose $\alpha_d \leqslant 340/531$. If $h_{d+1} = \cdots = h_k = 0$, then by Proposition 3.3 we get

$$\begin{aligned} U(h) &\ll Y^{1-\min((1-\alpha_d)/3,\, 1/6,\, \alpha_d/2)} \log^{4.5} Y \\ &\ll Y^{1-\min(\alpha_k/2,\, 191/1593)} \log^{4.5} Y \\ &\ll Y^{1-\delta_1} \log^{4.5} Y. \end{aligned}$$

If there is at least one $h_j \neq 0$ $(j>d)$, then $d \leqslant k-1$. By Proposition 3.2 we have

$$U(h) \ll Y^{1-\min((1-\alpha_d)/3,\, \alpha_d-\alpha_{d+1},\, \alpha_d/4)} \log^{5.5} Y.$$

If $\alpha_d - \alpha_{d+1} \leqslant \alpha_d/4$, then

$$\min((1-\alpha_d)/3, \alpha_d - \alpha_{d+1}, \alpha_d/4) = \min((1-\alpha_d)/3, \alpha_d - \alpha_{d+1}).$$

If $\alpha_d - \alpha_{d+1} > \alpha_d/4$, then

$$\alpha_d/4 \geqslant \alpha_{d+1}/3 \geqslant \alpha_k/3.$$

So we have

$$\begin{aligned} U(h) &\ll Y^{1-\min((1-\alpha_d)/3,\, \alpha_d - \alpha_{d+1},\, \alpha_d/4)} \log^{5.5} Y \\ &\ll Y^{1-\min((1-\alpha_d)/3,\, \alpha_d - \alpha_{d+1},\, \alpha_k/3)} \log^{5.5} Y \\ &\ll Y^{1-\delta_1} \log^{5.5} Y. \end{aligned}$$

This completes the proof of Theorem 1.

Using Proposition 3.1* in the proof of Theorem 1 we get Theorem 2.

5. EXPONENTIAL SUMS OVER PRIMES

Suppose $l \geqslant 2$ is a fixed integer; $1 > \gamma_1 > \gamma_2 > \cdots > \gamma_l > 0$ are real numbers; Y is a large positive number; and $0 < \delta = \delta(\gamma_1) < 1/2$ is a constant depending only on γ_1. The aim of this section is to estimate the exponential sum

$$S(Y; h_1, ..., h_l, \gamma_1, ..., \gamma_l) = \sum_{Y < n \leqslant 2Y} \Lambda(n)\, e\left(\sum_{j=1}^{l} h_j n^{\gamma_j} \right),$$

where h_j are real numbers such that $1 \leqslant |h_j| \leqslant Y^{\delta(\gamma_1)}$, $j = 1, 2, ..., l$. In this section we always set $R = \sum_{j=1}^{l} |h_j|\, Y^{\gamma_j}$.

Lemma 5.1. *Let* $\delta = \delta(\gamma_1) = \min(\gamma_1/(4l-2), (1-\gamma_1)/3)$ *and* a_m *be a sequence of complex numbers such that*

$$\sum_{M < m \leqslant 2M} |a_m|^2 \ll M \log^{2A} M, \qquad A > 0.$$

Then for $M \ll Y^{1-\delta} R^{-1/2}$, $MN = Y$, *we have*

$$S_{\mathrm{I}} = \sum_{m \sim M} a(m) \sum_{n \sim Y/m} e\left(\sum_{j=1}^{l} h_j (mn)^{\gamma_j} \right) \ll Y^{1-\delta} \log^{A} Y. \tag{5.1}$$

Proof. By Lemma 2.9 we have

$$\begin{aligned} S_{\mathrm{I}} &\ll \sum_{M<m\leqslant 2M} |a_m|\,(R^{1/2}+NR^{-1/(k+1)}) \\ &\ll MR^{1/2}\log^A Y + Y^{1-(\gamma_1/(k+1))}\log^A Y. \end{aligned}$$

Whence Lemma 5.1 follows.

LEMMA 5.2. *Let* $\delta=\delta(\gamma_1)=\min(\gamma_1/(4l-2), (1-\gamma_1)/3)$ *and* $a(m)$ *and* $b(n)$ *be complex numbers such that*

$$\sum_{M<m\leqslant 2M} |a(m)|^2 \ll M\log^{2A} M,$$

$$\sum_{N<m\leqslant 2N} |b(n)|^2 \ll N\log^{2B} N, \qquad A>0, \quad B>0.$$

Then for $Y^{2\delta}\ll N\ll RY^{-2(l-1)\,\delta}$, $MN=Y$, *we have*

$$\begin{aligned} S_{\mathrm{II}} &= \sum_{m\sim M} a(m)\sum_{n\sim N} b(n)\, e\left(\sum_{j=1}^{l} h_j (mn)^{\gamma_j}\right) \\ &\ll Y^{1-\delta}\log^{A+B+1} Y. \end{aligned} \tag{5.2}$$

Proof. Take $Q=[Y^{2\delta}\log^{-1} Y]$, then $10<Q=o(N)$. By Cauchy's inequality and Lemma 2.3 we get

$$|S_{\mathrm{II}}|^2 \ll \frac{M^2N^2\log^{2A+2B} Y}{Q} + \frac{MN\log^{2A} Y}{Q}\sum_{q=1}^{Q} E_q, \tag{5.3}$$

with

$$E_q=\sum_{n\sim N} |b(n+q)\,b(n)|\left|\sum_{m\sim M} e(f(m,n,q))\right|,$$

where

$$f(m,n,q)=\sum_{j=1}^{l} h_j m^{\gamma_j}\Delta(n,q;\gamma_j)$$

and $\Delta(n,q;\gamma_j)$ is defined in the same way as in Section 3.

For each fixed q, we find that

$$\begin{aligned}\sum_{j=1}^{l} |h_j m^{\gamma_j} \Delta(n, q; \gamma_j)| &= \sum_{j=1}^{l} |h_j|\, m^{\gamma_j}(\gamma_j q n^{\gamma_j - 1} + O(q^2 N^{\gamma_j - 2})) \\ &\asymp \frac{Rq}{N} = R^*(q),\end{aligned}$$

say. Since

$$\frac{R^*(q)}{N} \ll \frac{Rq}{MN} \ll \frac{Y^{\delta+\gamma_1} q}{MN} \ll \frac{Y^{\delta+\gamma_1+2\delta}}{MN \log Y} \ll \log^{-1} Y,$$

by Lemma 2.8 we get

$$\begin{aligned}\sum_{q=1}^{Q} \sum_{n \sim N} |b(n+q)\, b(n)| \left| \sum_{m \sim M} e(f(m, n, q)) \right| &\ll \sum_{q=1}^{Q} \sum_{n \sim N} |b(n+q)\, b(n)|\; M(R^*(q))^{-1/l} \\ &\ll M N^{1+1/l} Q^{1-1/l} R^{-1/l} \log^{2B} Y \ll Y \log^{2B} Y, \qquad (5.4)\end{aligned}$$

where in the last step we used the assumption that $N \ll R Y^{-2(l-1)\delta}$. So Lemma 5.2 follows from (5.3) and (5.4).

LEMMA 5.3. *The estimate*

$$S(Y; h_1, ..., h_l, \gamma_1, ..., \gamma_l) \ll Y^{1-\min((1-\gamma_1)/3,\, \gamma_1/(4l-2))} \log^{5.5} Y. \qquad (5.5)$$

Proof. We only sketch the proof. We begin with (3.15) by choosing $u = Y^{2\delta}$, $v = Y^{1+2(l-1)\delta} R^{-1}$. We use Lemma 2.8 to estimate S_1 and Lemma 5.2 to estimate S_3. We use Lemma 5.1 to estimate S_2 for $M \ll Y^{1-\delta} R^{-1/2}$ and Lemma 5.2 to estimate S_2 for $Y^{1-\delta} R^{-1/2} \ll M \ll uv$.

LEMMA 5.4. *Suppose* $1/2 < \gamma_1 < 1$, $\delta = \delta(\gamma_1) = \min((3-2\gamma_1)/14,\ (\gamma_1 - 1/2)/2l)$. *Let* $a(m)$ *and* $b(n)$ *be complex numbers such that*

$$\sum_{M < m \leqslant 2M} |a(m)|^2 \ll M \log^{2A} M,$$

$$\sum_{N < m \leqslant 2N} |b(n)|^2 \ll N \log^{2B} N, \qquad A > 0, \quad B > 0.$$

Then for $Y^{2\delta} \ll N \ll Y^{1/2}$, $MN \sim Y$, *we have*

$$S_{\mathrm{II}} = \sum_{m \sim M} a(m) \sum_{n \sim N} b(n)\, e\left(\sum_{j=1}^{l} h_j (mn)^{\gamma_j}\right)$$
$$\ll Y^{1-\delta} \log^{A+B+1} Y. \tag{5.6}$$

Proof. Using Lemma 2.7 to estimate the right side of (5.3) we get

$$\sum_{q=1}^{Q} \sum_{n \sim N} |b(n+q)\, b(n)| \left| \sum_{m \sim M} e(f(m, n, q)) \right|$$
$$\ll \sum_{q=1}^{Q} \sum_{n \sim N} |b(n+q)\, b(n)|$$
$$\times ((Rq)^{1/2} N^{-1/2} + M(Rq)^{-1/(l+1)} N^{1/(k+1)})$$
$$\ll (N^{1/2} R^{1/2} Q^{3/2} + MN^{1+1/(l+1)} Q^{l/(l+1)} R^{-1/(l+1)}) \log^{2B} Y$$
$$\ll Y \log^{2B} Y.$$

Whence Lemma 5.4 follows.

Lemma 5.5. *Suppose* $1/2 < \gamma_1 < 1$, $\delta = \delta(\gamma_1) = \min((3-2\gamma_1)/14,\ (\gamma_1 - 1/2)/2l)$. *Then the estimate*

$$S(Y; h_1, ..., h_l, \gamma_1, ..., \gamma_l) \ll Y^{1-\delta} \log^{5.5} Y. \tag{5.7}$$

Proof. We sketch the proof. Take $u = Y^{2\delta}$ and $v = Y^{1/2}$ in (3.15). We use Lemma 2.8 to estimate S_1 and S_2 for $m \leqslant Y^{2\delta}$. We use Lemma 5.4 to estimate S_3 and S_2 for $m > Y^{2\delta}$.

Proposition 5.1. *Let* $\delta = \min(\gamma_1/(4l-2),\ 1/(4l+6))$. *Then we have*

$$S(Y; h_1, ..., h_l, \gamma_1, ..., \gamma_l) \ll Y^{1-\delta} \log^{5.5} Y. \tag{5.8}$$

Proof. This estimate follows from Lemma 5.3 for $\gamma_1 \geqslant (2l+3/2)/(2l+3)$ and from Lemma 5.5 for $\gamma_1 \leqslant (2l+3/2)/(2l+3)$.

6. PROOF OF THEOREM 3

Following the proof of Theorem 1, we only need to estimate $U(h)$ for fixed $h = (h_1, ..., h_k) \neq (0, ..., 0)$. We take $H = Y^{\delta_3}$, where δ_3 is defined in Section 1.

For a fixed $h=(h_1, ..., h_k)\neq(0, ..., 0)$, let $n_0(h)$ denote the number of h_j such that $h_j\neq 0$, and let d be the first integer j with $h_j\neq 0$. If $n_0(h)\geqslant 2$, then by Proposition 5.1 we have

$$\begin{aligned} U(h) &\ll Y^{1-\min(1/(4n_0(h)+6),\ \alpha_d/(4n_0(h)-2))}\log^{5.5} Y \\ &\ll Y^{1-\min(1/(4k+6),\ \alpha_k/(4k-2))}\log^{5.5} Y. \end{aligned}$$

Now suppose $n_0(h)=1$. If $\alpha_d\geqslant 340/531$, then by Proposition 3.1* we have

$$U(h)\ll Y^{1-40/407}\log^{11.5} Y\ll Y^{1-\delta_3}\log^{5.5} Y.$$

If $\alpha_d<340/531$, then by Proposition 3.3 we have

$$\begin{aligned} U(h) &\ll Y^{1-\min((1-\alpha_d)/3,\ 1/6,\ \alpha_d/2)}\log^{4.5} Y \\ &\ll Y^{1-\delta_3}\log^{5.5} Y. \end{aligned}$$

This completes the proof of Theorem 3.

REFERENCES

1. R. C. Baker and G. Kolesnik, On the distribution of p^α modulo one, *J. Reine Angew. Math.* **356** (1985), 174–193.
2. A. Balog, On the fractional part of p^θ, *Arch. Math.* **40** (1983), 434–440.
3. A. Balog, On the distribution of $p^\theta \pmod 1$, *Acta. Math. Hungar.* **45**, Nos. 1–2 (1985), 179–199.
4. E. Bombieri and H. Iwaniec, On the order of $\zeta(\frac{1}{2}+it)$, *Ann. Scuola Norm. Sup. Pisa* **13**, No. 3 (1986), 449–472.
5. E. Fouvry and H. Iwaniec, Exponential sums for monomials, *J. Number Theory* **33** (1989), 311–333.
6. S. W. Graham and G. Kolesnik, "Van Der Corput's Method of Exponential Sums," Cambridge Univ. Press, Cambridge, UK, 1991.
7. S. A. Gritsenko, On a problem of I. M. Vinogradov, *Mat. Zametki* **39** (1986), 625–650. [In Russian]
8. G. Harman, On the distribution of $\sqrt{p}$ modulo one, *Mathematika* **30** (1983), 104–110.
9. D. R. Heath-Brown, The Pjatecskil–Sapiro prime number theory, *J. Number Theory* **16** (1983), 242–266.
10. R. M. Kaufman, The distribution of $\sqrt{p}$, *Mat. Zametki* **26** (1979), 497–504.
11. J. Schoissengeier, The connection between the zeros of the ζ-function and sequences $(g(p))$, p prime mod 1, *Mh. Math.* **87** (1979), 21–52.
12. J. Schoissengeier, Eine neue Diskrepanz fur gewisse Primzahlfolgen, *in* "Sitzungsber. Math. Natur. Kl.," Abt. Ii, 187. Bd., 4. bis 7. Heft, 1978.
13. D. I. Tolev, On the simultaneous distribution of the fractional parts of different powers of primes numbers, *J. Number Theory* **37** (1991), 298–306.

14. R. C. Vaughan, An elementary method in prime number theory, *Acta Arithmetica* **37** (1980), 111–115.
15. I. M. Vinogradov, "Special Variants of the Method of Trigonometric Sums," Izda. Nauka, Moskow, 1976. [In Russian]
16. W. Zhai, On the k-dimensional Piatetski–Shapiro prime number theorem, *Sci. in China* (*Ser. A*) **29** (1999), 797–806.